FINAL PROJECT

CEE 361/351: Introduction to Finite Element Methods

Due: Monday January 15th @ Midnight

Instructions

Submit the final report and code as a single *pdf file in dropbox on Blackboard.

CEE 361: You may work on this project in groups of up to three people. You should not compare results with other groups. Each member of the group should submit a project report typed in LaTeX.

CEE 513: You are to work individually and should not compare results. You should submit a project report typed in LaTeX.

The final report should contain:

1. "Introduction" section that clearly describes the problem
2. "Mathematical Formulation" section that formulates the boundary value problem with all relevant problem data (eg. material constants, dimensions etc . . .)
3. "Numerical Discretization" section that discusses the Galerkin approximation, the matrix form, the choices of finite element spaces (eg. linear vs quadratic etc) for each of the fields (the functions we are solving for, namely the displacement and the pressures of the porous medium)
4. "Implementation" section that describes the implementation in FEniCS, the particular choices of the meshes, possible refinements, etc. . . .
5. "Design Methodology" section that describes how you approached finding the optimal design
6. "Results" section that presents the results of the simulations
7. "Discussion" section

Problem Description

You need to design the width ℓ_b of the dam that rests on a soil that is saturated with water. The soil deforms when subjected to loads (the weight of the dam) and swells with increased pore pressure. For simplicity, we assume that the weight of the dam does not change as ℓ_b changes (an idealization) and the weight equals 2.5×10^5 N $(= W)$. The weight of the dam transfers to the underlying soil as uniformly distributed load over the length ℓ_b, and equals

$$\boldsymbol{t} = -\gamma \mathbf{e}_2 = \frac{W}{\ell_b}\mathbf{e}_2.$$

The height of water on the upstream side of the dam is 4.5 m $(= h_w)$ and is 0 m on the downstream side. This creates a pressure gradient in the soil.

You should provide the optimal design dimension ℓ_b such that the maximum vertical deformation under the dam is less then $5.0mm$ and no point experiences uplift. Namely you want to find the *minimum* value of ℓ_b that satisfies the displacement constraint.

Although the soil domain in reality is a semi-infinite continuum, we will only consider a sub-domain large enough of size $\ell_s = 20$ m and height $h_s = 15$ m. Figure 1 showcases the problem to be considered.

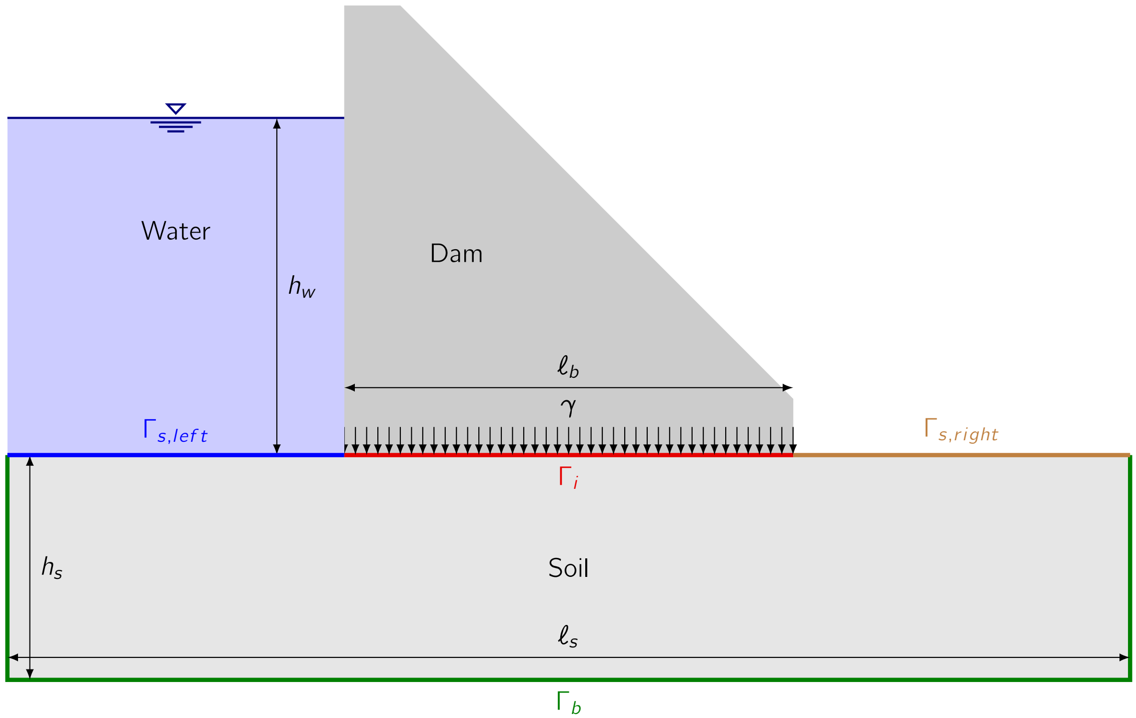

Figure 1: The schematic of the problem

The deformable porous material

We are interested in seeing how a fluid saturated porous solid deforms. You may think of a porous material as a sponge, for example. As you squeeze the sponge, the fluid has to travel through the pores to exit the sponge and in the process it hinders the deformation of the sponge. Depending on how quickly the fluid can escape the sponge, the faster the sponge will deform.

What controls the speed at which a fluid escapes is the conductivity κ of the medium and the gradient of the pressure p in the pores. This relation is known as Darcy's law and (with some simplifications) it states that the relative velocity $\overline{\boldsymbol{v}}$ of the fluid with respect to the solid is given by

$$\overline{\boldsymbol{v}}(\nabla p) = -\kappa \nabla p$$

where p is the pore pressure.

As now we have a composite system that carries the load, namely both the fluid and the solid can carry stresses, we have that the (total) stress tensor $\boldsymbol{\sigma}$ is defined as the sum of the stress carried by the solid $\boldsymbol{\sigma}'$ and the stress carried by the fluid $-p\mathbf{1}$. Thus the equation of balance of momentum that dictates how the solid will deform becomes

$$\nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}, p) = \boldsymbol{f} \tag{1}$$

where, as per usual, $\boldsymbol{u}$ is the deformation of the solid, p is the pore pressure, and $\boldsymbol{f}$ is some body force (eg. gravity) and

$$\boldsymbol{\sigma}(\nabla \boldsymbol{u}, p) = \boldsymbol{\sigma}'(\nabla \boldsymbol{u}) - p\mathbf{1}$$

and

$$\boldsymbol{\sigma}'(\nabla \boldsymbol{u}) = \lambda \operatorname{tr}(\nabla \boldsymbol{u})\mathbf{1} + 2\mu\, \varepsilon(\nabla \boldsymbol{u}), \quad \varepsilon(\nabla \boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top}).$$

In addition to balance of momentum we now have to ensure that mass is also conserved and mass conservation is governed by the equation

$$\nabla \cdot \overline{\boldsymbol{v}}(\nabla p) = -\kappa \Delta p = 0 \tag{2}$$

where $\nabla \cdot \overline{\boldsymbol{v}}$ represents the net flux of pore water, and since we are conserving mass it should be zero (note that, as always, $\Delta \bullet = \nabla \cdot \nabla \bullet$).

Project task

With the above in mind your goal is to find the minimum width ℓ_b such that $|\boldsymbol{u} \cdot \mathbf{e}_2| \leq 5.0$ mm and $\boldsymbol{u} \cdot \mathbf{e}_2 < 0$ for every point $\boldsymbol{x}$ underneath the dam.

The strong form of the problem statement is:

Find $\boldsymbol{u}$ and p such that

$$\nabla \cdot \boldsymbol{\sigma}(\nabla \boldsymbol{u}, p) = 0, \qquad \forall \boldsymbol{x} \in \Omega, \tag{3}$$

$$\kappa \Delta p = 0, \qquad \forall \boldsymbol{x} \in \Omega, \tag{4}$$

and boundary conditions

$$\boldsymbol{\sigma n} = -\gamma \boldsymbol{n} \qquad \forall \boldsymbol{x} \in \Gamma_i \tag{5}$$

$$\boldsymbol{\sigma n} = \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \Gamma_{s,left} \& \Gamma_{s,right} \tag{6}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \Gamma_b \tag{7}$$

$$p = -\rho_w g h_w \qquad \forall \boldsymbol{x} \in \Gamma_{s,left}. \tag{8}$$

$$p = 0 \qquad \forall \boldsymbol{x} \in \Gamma_{s,right}. \tag{9}$$

and

$$\boldsymbol{\sigma}(\nabla \boldsymbol{u}, p) = \boldsymbol{\sigma}'(\nabla \boldsymbol{u}) - p\mathbf{1}$$

and

$$\boldsymbol{\sigma}'(\nabla \boldsymbol{u}) = \lambda \operatorname{tr}(\nabla \boldsymbol{u})\mathbf{1} + 2\mu\, \varepsilon(\nabla \boldsymbol{u}), \quad \varepsilon(\nabla \boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top).$$

Material Constants

For the purpose of this project, use the following material constants.

Property	Value	Units
Youngs' Modulus (E)	30	MPa
Poisson's Ratio (ν)	0.4	-
Conductivity (κ)	4.0×10^{-6}	m/s
Density of water (ρ_w)	1000	kg/m^3