HOMEWORK 1 - SOLUTIONS

CEE 361-513: Introduction to Finite Element Methods

Due: Friday Sept. 29

NB: Students taking CEE 513 must complete all problems. All other students will not be graded for problems marked with $\star$, but are encourage to attempt them anyhow.

PROBLEM 1

Unless otherwise specified, you may assume that $\{\mathbf{e}_i\}_{i=1}^d$ is a set of orthonormal basis associated with a set of cartesian coordinates $\{x_i\}_{i=1}^d$ (cf. the figure on the right). Use indicial notation when appropriate.

1. Show that for two vectors $\boldsymbol{a}, \boldsymbol{b}$ the following holds $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$.

Solution :

Let $\boldsymbol{a} = a_i \mathbf{e}_i$ and $\boldsymbol{b} = b_j \mathbf{e}_j$

$$\begin{aligned}\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= a_i \mathbf{e}_i \cdot (a_j \mathbf{e}_j \times b_k \mathbf{e}_k) \\ &= a_i a_j b_k \epsilon_{jk\ell} \mathbf{e}_i \cdot \mathbf{e}_\ell = a_i a_j b_k \epsilon_{jk\ell} \delta_{i\ell} = a_i a_j b_k \epsilon_{jki}\end{aligned}$$

Note that $\epsilon_{jki} = -\epsilon_{ikj}$ hence

$$a_i a_j b_k \epsilon_{jki} = -a_i a_j b_k \epsilon_{ikj}$$

Also, since i, j are just dummy indeces

$$a_i a_j b_k \epsilon_{jki} = a_i a_j b_k \epsilon_{ikj}$$

thus we have that

$$-a_i a_j b_k \epsilon_{ikj} = a_i a_j b_k \epsilon_{ikj} \Rightarrow a_i a_j b_k \epsilon_{ikj} = 0.$$

2. Let $d = 3$ and $\boldsymbol{u}(\boldsymbol{x}) = x_1 x_2 x_3 \mathbf{e}_1 + x_1 \mathbf{e}_2 + x_1 \mathbf{e}_3$ compute $\nabla \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$.

Solution :

$$\begin{aligned}\nabla \boldsymbol{u} &= \frac{\partial \boldsymbol{u}}{\partial x_j} \otimes \mathbf{e}_j \\ &= x_2 x_3 \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_1 + x_1 x_3 \mathbf{e}_1 \otimes \mathbf{e}_2 + x_1 x_2 \mathbf{e}_1 \otimes \mathbf{e}_3 \\ \nabla \cdot \boldsymbol{u} &= \frac{\partial \boldsymbol{u}}{\partial x_j} \cdot \mathbf{e}_j \\ &= x_2 x_3\end{aligned}$$

3. Let $d = 2$, $\boldsymbol{u}(\boldsymbol{x}) = x_1 x_2 \mathbf{e}_1 + x_1 \mathbf{e}_2$, and $\boldsymbol{v}(\boldsymbol{x}) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$. If $\boldsymbol{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{u} \otimes \boldsymbol{v}$, what are the values of T_{ij}.

Solution :

$$T_{ij} = \mathbf{e}_i \cdot T \cdot \mathbf{e}_j$$

$$T = \begin{bmatrix} x_1^2 x_2 & x_1 x_2^2 \\ x_1^2 & x_1 x_2 \end{bmatrix}$$

4. What is the value of $\boldsymbol{I} : \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity tensor.

Solution :

$$\begin{aligned}\boldsymbol{I} : \boldsymbol{I} &= \delta_{ij}\delta_{ij} \\ &= 3\end{aligned}$$

5. Let $\boldsymbol{u}$ be a vector. Is $\boldsymbol{T}(\boldsymbol{u}) = \exp(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_1$ a tensor? Show why or why not.

Solution :

$$\begin{aligned}\boldsymbol{T}(\alpha\boldsymbol{u}) &= \exp(\alpha\boldsymbol{u} \cdot \mathbf{e}_1)\,\mathbf{e}_1 \\ &\neq \alpha\exp(\boldsymbol{u} \cdot \mathbf{e}_1)\,\mathbf{e}_1\end{aligned}$$

Hence, not a tensor.

6. Let $\boldsymbol{u}$ be a vector. Is $\boldsymbol{T}(\boldsymbol{u}) = 10\,(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1 + (\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2$ a tensor? Show why or why not.

Solution :
Now,

$$\begin{aligned}\boldsymbol{T}(\alpha\boldsymbol{u}) &= 10\,(\alpha\boldsymbol{u} \cdot \mathbf{e}_2)\,\mathbf{e}_1 + (\alpha\boldsymbol{u} \cdot \mathbf{e}_1)\,\mathbf{e}_2 \\ &= \alpha\left[10\,(\boldsymbol{u} \cdot \mathbf{e}_2)\,\mathbf{e}_1 + (\boldsymbol{u} \cdot \mathbf{e}_1)\,\mathbf{e}_2\right] \\ &= \alpha\boldsymbol{T}(\boldsymbol{u})\end{aligned}$$

Also,

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{v} + \boldsymbol{u}) &= 10\,((\boldsymbol{v} + \boldsymbol{u}) \cdot \mathbf{e}_2)\,\mathbf{e}_1 + ((\boldsymbol{v} + \boldsymbol{u}) \cdot \mathbf{e}_1)\,\mathbf{e}_2 \\ &= 10\,(\boldsymbol{v} \cdot \mathbf{e}_2)\,\mathbf{e}_1 + (\boldsymbol{v} \cdot \mathbf{e}_1)\,\mathbf{e}_2 + 10\,(\boldsymbol{u} \cdot \mathbf{e}_2)\,\mathbf{e}_1 + (\boldsymbol{u} \cdot \mathbf{e}_1)\,\mathbf{e}_2 \\ &= \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})\end{aligned}$$

Hence, a tensor.

7. $\star$ Show that $(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{A}) = \boldsymbol{u} \otimes \boldsymbol{A}^\top\boldsymbol{v}$.

Solution :

$$\begin{aligned}(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{A}) &= (u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j) \cdot (A_{kl}\mathbf{e}_k \otimes \mathbf{e}_l) \\ &= u_i v_j A_{kl}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_l) \\ &= u_i v_j A_{kl}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_l \\ &= u_i v_j A_{jl}\mathbf{e}_i \otimes \mathbf{e}_l \\ &= u_i\mathbf{e}_i \otimes \left(A^\top_{jl} v_j \mathbf{e}_l\right) \\ &= \boldsymbol{u} \otimes \left(\boldsymbol{A}^\top\boldsymbol{v}\right)\end{aligned}$$

8. $\star$ Show that $\nabla \cdot (\psi\boldsymbol{u}) = \nabla\psi \cdot \boldsymbol{u} + \psi\nabla \cdot \boldsymbol{u}$ for $\boldsymbol{u} \in \mathbb{R}^d, \psi \in \mathbb{R}$.

Solution :

$$\begin{aligned}\nabla \cdot (\psi \boldsymbol{u}) &= \frac{\partial \psi u_i \mathbf{e}_i}{\partial x_j} \cdot \mathbf{e}_j \\ &= \frac{\partial \psi}{\partial x_j} \mathbf{e}_j \cdot u_i \mathbf{e}_i + \psi \frac{\partial u_i \mathbf{e}_i}{\partial x_j} \cdot \mathbf{e}_j \\ &= \nabla \psi \cdot \boldsymbol{u} + \psi \nabla \cdot \boldsymbol{u}\end{aligned}$$

9. $\star$ Show that $\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = \nabla \boldsymbol{u}\, \boldsymbol{v} + \boldsymbol{u} \nabla \cdot \boldsymbol{v}$.

Solution :

$$\begin{aligned}\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) &= \frac{\partial \left(u_i \mathbf{e}_i \otimes v_j \mathbf{e}_j \right)}{\partial x_k} \cdot \mathbf{e}_k \\ &= \frac{\partial \left(u_i \mathbf{e}_i \otimes v_j \mathbf{e}_j \right)}{\partial x_k} \cdot \mathbf{e}_k \\ &= \frac{\partial u_i \mathbf{e}_i}{\partial x_k} \otimes \mathbf{e}_k v_j \mathbf{e}_j + \frac{\partial v_j}{\partial x_j} u_i \mathbf{e}_i \\ &= \frac{\partial u_i \mathbf{e}_i}{\partial x_k} \otimes \mathbf{e}_k v_j \mathbf{e}_j + \boldsymbol{u} \nabla \cdot \boldsymbol{v} \\ &= \nabla \boldsymbol{u} \boldsymbol{v} + \boldsymbol{u} \nabla \cdot \boldsymbol{v}\end{aligned}$$

PROBLEM 2

To practice with `Python` do the following operations

1. Let $\boldsymbol{u} = 1\mathbf{e}_1 + 2\mathbf{e}_2$. Construct a *unit* vector $\boldsymbol{n}$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. (Hint: create any vector $\boldsymbol{v}$ that is not linearly dependent with $\boldsymbol{u}$, then let $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{u}/\|\boldsymbol{u}\|^2\, \boldsymbol{u}$ and then let $\boldsymbol{n} = \boldsymbol{w}/\|\boldsymbol{w}\|$).

```
## Python Code :
# define u
u = np.array([1.0,2.0])

# Using hint
v = np.array([4.0,2.0])
norm_u = LA.norm(u)
w = v - u*np.dot(u,v)/(norm_u**2)
w_norm = LA.norm(w)
n1 = w/w_norm

print(n1)
# check the dot product
print(np.dot(u,n1))
```

2. Let $\boldsymbol{u} = 3\mathbf{e}_1 + 2\mathbf{e}_2 + 4\mathbf{e}_3$, $\boldsymbol{v} = 5\mathbf{e}_1 + 1\mathbf{e}_2 + 4\mathbf{e}_3$. Construct a *unit* vector $\boldsymbol{n}$ that is orthogonal to $\boldsymbol{u}, \boldsymbol{v}$. (Hint: $\times$)

```
u = np.array([3.0,2.0,4.0])
v = np.array([5.0,1.0,4.0])

w = np.cross(u,v)
norm_w = LA.norm(w)

n3 = w/norm_w
#check
print(np.dot(u,n3))
print(np.dot(v,n3))
```

3. Given two points $\boldsymbol{x}_a = 1\mathbf{e}_1 + 2\mathbf{e}_2$, $\boldsymbol{x}_b = 5\mathbf{e}_1 + 7\mathbf{e}_2$, construct a tensor $\boldsymbol{T}$ that projects vectors along the direction of $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$. Remember that a projection must satisfy $\boldsymbol{T}(\boldsymbol{T}(\boldsymbol{b})) = \boldsymbol{T}(\boldsymbol{b})$ for all vectors $\boldsymbol{b}$.

```
x_a = np.array([1.0,2.0])
x_b= np.array([5.0,7.0])
a = x_b-x_a
norm_a = LA.norm(a)
n_a = a/norm_a
c = np.outer(n_a,n_a)

# Print the tensor
print(c)
# Check with a random vector whether T(T(b)) = T(b)
u = np.array([1.0,3.0])
print(np.dot(c,u))
print(np.dot(c,np.dot(c,u)))
```

4. ⋆ Given a function $f(\boldsymbol{x}) = sin(x_1)e^{x_2}$ derive ∇f and plot the vector field

```
#define symbolic vars, function
x,y=sp.symbols('x y')
fun=sp.sin(x)*sp.exp(y)

#take the gradient symbolically
gradfun=[sp.diff(fun,var) for var in (x,y)]

#turn into a bivariate lambda for numpy
numgradfun=sp.lambdify([x,y],gradfun,'numpy')
numfun=sp.lambdify([x,y],fun,'numpy')
X,Y=np.meshgrid(np.arange(-2.0*np.pi,2.0*np.pi,0.2),np.arange(-2.0,3.0,0.2))
graddat=numgradfun(X,Y)
fundat=numfun(X,Y)

fig, ax = plt.subplots()
hc=plt.contourf(X,Y,fundat,np.linspace(fundat.min(),fundat.max(),100))
ax.quiver(X,Y,graddat[0],graddat[1])
plt.colorbar(hc)
ax.set_title('Plot of gradient')
ax.set_xlabel('x-coordinates')
ax.set_ylabel('y-coordinates')
plt.show()
```

PROBLEM 3

1. Let u, v be sufficiently smooth functions of x. Show step-by-step that

$$\int_0^\ell \left[\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)\right] v\,dx = \int_0^\ell EI\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx + \left[\frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)v\right]\Bigg|_0^\ell - \left[EI\frac{d^2u}{dx^2}\frac{dv}{dx}\right]\Bigg|_0^\ell$$

where E, I are constants.

Solution :

Use integration by parts

$$\begin{aligned}
&\int_0^\ell \left[\frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right)\right] v \;\; dx \\
&= \left[\frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)v\right]\Bigg|_0^\ell - \int_0^\ell \left[\frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)\right]\frac{dv}{dx}\;\; dx \\
&= \left[\frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)v\right]\Bigg|_0^\ell - \left[EI\frac{d^2u}{dx^2}\frac{dv}{dx}\right]\Bigg|_0^\ell + \int_0^\ell \left(EI\frac{d^2u}{dx^2}\right)\frac{d^2v}{dx^2}\;\; dx
\end{aligned}$$

2. $\star$ Let $\boldsymbol{\sigma}(\boldsymbol{x}) \in \mathbb{R}^d \times \mathbb{R}^d, \boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, and $\boldsymbol{\eta}(\boldsymbol{x}) \in \mathbb{R}^d$ (with both $\boldsymbol{\sigma}$ and $\boldsymbol{\eta}$ being integrable and sufficiently smooth), show that

$$\int_\Omega (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\eta}dV = \int_{\partial\Omega}\boldsymbol{\eta}\cdot\boldsymbol{\sigma}\boldsymbol{n}dS - \int_\Omega \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}dV.$$

Solution :

$$\begin{aligned}
\nabla\cdot(\boldsymbol{\sigma}\boldsymbol{\eta}) &= \frac{\partial\boldsymbol{\sigma}\boldsymbol{\eta}}{\partial x_i}\cdot\mathbf{e}_i \\
&= \frac{\partial\sigma_{ij}}{\partial x_j}\mathbf{e}_i\cdot\boldsymbol{\eta} + \mathbf{e}_i\cdot\boldsymbol{\sigma}\frac{\partial\boldsymbol{\eta}}{\partial x_i} \\
&= (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\eta} + \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}
\end{aligned}$$

Rearranging and subtituting in the question:

$$\int_\Omega (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\eta}dV = \int_\Omega \nabla\cdot(\boldsymbol{\sigma}\boldsymbol{\eta})\,dV - \int_\Omega \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}dV$$

Using the Gauss' theorem

$$\int_\Omega (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\eta}dV = \int_{\partial\Omega} (\boldsymbol{\sigma}\boldsymbol{\eta})\cdot\boldsymbol{n}dS - \int_\Omega \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}dV$$

Since $\boldsymbol{\sigma}$ is symmetric

$$\int_\Omega (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\eta}dV = \int_{\partial\Omega} \boldsymbol{\eta}\cdot(\boldsymbol{\sigma}\boldsymbol{n})\,dS - \int_\Omega \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}dV$$