HOMEWORK 2

CEE 361-513: Introduction to Finite Element Methods

Due: October 12, 2018

NB: Students taking CEE 513 must complete all problems. All other students will not be graded for problems marked with $\star$, but are encourage to attempt them anyhow.

PROBLEM 1

Write down the expression for the following famous PDEs (for two-dimensional problems) and provide a short description of what they represent

1. Heat equation

   Solution :

   $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

   where :

   $$T : \text{Temperature}$$
   $$\alpha : \text{thermal diffusivity}$$

   It represents how the temperature changes at a point as the heat spreads through space over time.

2. Euler-Bernoulli equations

   Solution :

   $$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q(x)$$

   where:

   $$E : \text{Youngs modulus}$$
   $$I : \text{Moment of opportunity}$$
   $$w : \text{vertical displacement of the neutral axis}$$

   The equation describes the relationship between the deflection of neutral axis of the beam and the load applied.

3. Navier-Stokes equations

Solution :

The following equations together are in generally referred to as the Navier-Stokes equation:

$$
\begin{aligned}
\text{Continuity} &: \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\text{Momentum} &: \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \nu \Delta \boldsymbol{u} - \frac{\nabla P}{\rho} + \boldsymbol{g} \\
\text{Energy} &: \frac{\partial s}{\partial t} = -\boldsymbol{u} \cdot \nabla s + \frac{Q}{T}
\end{aligned}
$$

where:

$$
\begin{aligned}
\rho &: \text{density} \\
\boldsymbol{u} &: \text{velocity} \\
\nu &: \text{kinematic viscosity} \\
P &: \text{Pressure} \\
\boldsymbol{g} &: \text{body acceleration} \\
s &: \text{entropy per unit mass} \\
Q &: \text{Heat transfered} \\
T &: \text{Temperature}
\end{aligned}
$$

The Navier-Stokes equation are the governing equations for fluid flow.

4. Cauchy momentum equations

Solution :

$$
\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{\sigma}) = \rho \boldsymbol{g}
$$

The equation describes the momentum transport in any continuum. The terms used in the equation represent the following:

$$
\begin{aligned}
\boldsymbol{u} &: \text{flow velocity} \\
\boldsymbol{\sigma} &: \text{stress tensor} \\
\boldsymbol{g} &: \text{body forces}
\end{aligned}
$$

5. Shallow water equations

Solution :

Under the assumption that there are no frictional, viscous and Coriolis forces:

$$\begin{aligned}
\frac{\partial \rho\eta}{\partial t} + \frac{\partial \rho\eta u}{\partial x} + \frac{\partial \rho\eta v}{\partial y} &= 0 \\
\frac{\partial \rho\eta u}{\partial t} + \frac{\partial}{\partial x}\left(\rho\eta u^2 + \frac{1}{2}\rho g\eta^2\right) + \frac{\partial \rho\eta uv}{\partial y} &= 0 \\
\frac{\partial \rho\eta v}{\partial t} + \frac{\partial}{\partial y}\left(\rho\eta v^2 + \frac{1}{2}\rho g\eta^2\right) + \frac{\partial \rho\eta uv}{\partial x} &= 0
\end{aligned}$$

where:

$$\begin{aligned}
\eta &: \text{Total fluid column height} \\
u, v &: \text{fluid's horizontal flow velocity averaged across the vertical section} \\
g &: \text{acceleration due to gravity} \\
\rho &: \text{fluid density}
\end{aligned}$$

They describe the flow of fluid below a pressure surface.

6. Kirchoff-Love plates' equation

Solution :

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 y^2} = \frac{q(x,y)}{k}$$

where:

$$\begin{aligned}
q(x,y) &= \text{transverse load} \\
k &= \text{bending stiffness}
\end{aligned}$$

The equation governs the bending of thin plates subject to external loads.

7. Transport equation

Solution :

$$\frac{\partial \rho c}{\partial t} = \nabla \cdot (\mathcal{D}\rho\nabla c) - \nabla \cdot (\vec{v}\rho c) + s$$

The equation describes the process by which particles, energy etc are tranported in a system. The terms in the equation represent:

$$\begin{aligned}
c &: Variable\ \ of\ \ interest \\
\rho &: Density \\
\mathcal{D} &: Diffusion\ \ Coeffiecient\ \ matrix \\
\vec{v} &: velocity\ \ field\ \ with\ \ which\ \ the\ \ quantity\ \ of\ \ interest\ \ is\ \ flowing \\
s &: Sources\ \ and\ \ sinks\ \ inthesystem
\end{aligned}$$

PROBLEM 2

1. Classify the the PDEs of Problem 1, namely specify the order and the linearity.

   Solution :

   (a) Heat equation : Order → 2; Linearity → Linear

   (b) Euler-Bernoulli equations : Order → 4; Linearity → Linear

   (c) Navier-Stokes equations : Order → 2; Linearity → Non-Linear

   (d) Cauchy momentum equations : Order → 2; Linearity → Non-Linear

   (e) Shallow water equations : Order → 1; Linearity → Non-Linear

   (f) Kirchoff-Love plates' equation : Order → 4; Linearity → Linear

   (g) Transport equation : Order → 2; Linearity → Linear

2. $\star$ Second order quasilinear[1] partial differential equations can be categorized as elliptic, parabolic, and hyperbolic. If we write the second order differential equation as

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i x_j} + (\text{lower order derivatives}) = 0$$

   then it can be categorized as

   Elliptic: if all the eigenvalues of the matrix a_{ij} have the same sign
   Parabolic: if one eigenvalue of the matrix a_{ij} is equal to zero and the rest have the same sign
   Hyperbolic: if all but one eigenvalues of the matrix a_{ij} have the same sign

   (a) Write down an explanation, in terms of characteristics, of what are the main differences between elliptic, parabolic, and hyperbolic PDEs.

[1] A quasilinear PDE is any PDE whose higher order derivatives are linear.

Solution :
The classification of the PDE is related to the characteristics of the PDE. The characteristics are (d-1) dimensional hypersurfaces in d-dimensional hyperspace. The prefix hyper- donote spaces of more than 3-dimension, e.g., including time along with the spatial dimensions. The characteristics, then, denote the path along which the information propagates in the solution domain. If the PDE has no real characteristic then there are no preferred path of information propagation. If the PDE has real characteristics, then information propagates along those characteristics.

Elliptic : No real characteristic curves.
Parabolic : One characteristic curve.
Hyperbolic : Two characteristic curces

This influence the properties of each type of problem.
Elliptic: Since these equations have no characteristic curves, hence no preferred path of information propagation, the solution at every point depends on the solution of the every other point.They generally govern the equilibrium problems.

Parabolic : Since they have one real characteristics, they have a preferred path for information propagation and hence govern propagation problems. They generally govern time marching problems.

Hyperbolic : They have two real characteristics and hence also govern propagation problems. They generally govern time marching problems and also allow for shocks to develop.

(b) For the PDEs of Problem 1, if they are second order and quasilinear, state whether they are elliptic, parabolic or hyperbolic.

Solution :

i. Heat Equation :

$$\text{d} = 2$$

We can write the equation as:

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$$

Hence the coefficients a_{ij} are:

$$a_{11} = -\alpha$$
$$a_{22} = a_{12} = a_{21} = 0$$

The matrix created with a_{ij} is

$$M = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues for this matrix are:

$$\lambda_1 = -\alpha$$
$$\lambda_2 = 0$$

Hence the equation is parabolic

Solution continued :

ii. Navier Stokes Equation : It is quasi-linear as the higher-order derivatives are linear.

$$\mathrm{d} = 4$$

Re-writing the momentum equation we get:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \frac{\nabla P}{\rho} - \boldsymbol{g} = 0$$

Hence the coefficients a_{ij} are:

$$\begin{aligned} a_{11} &= a_{12} = a_{13} = -\nu \\ a_{21} &= a_{22} = a_{23} = -\nu \\ a_{31} &= a_{32} = a_{33} = -\nu \\ a_{44} &= 0 = a_{42} = a_{43} \end{aligned}$$

The matrix created with a_{ij} is

$$M = \begin{bmatrix} -\nu & -\nu & -\nu & 0 \\ -\nu & -\nu & -\nu & 0 \\ -\nu & -\nu & -\nu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues for this matrix are:

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_2 &= 0 \\ \lambda_4 &= -3\nu \end{aligned}$$

Hence the equation is parabolic

iii. Cauchy-Momentum equation : It is quasi-linear as the higher order terms are linear. In the non-steady state case, it is the same as the momentum equation of the Navier-Stokes equation. Hence, it is parabolic.

Solution continued :

iv. Transport Equation :

$$\mathrm{d} = 4$$

Re-writing we get:

$$\frac{\partial \rho c}{\partial t} - \nabla \cdot (\mathcal{D}\rho\nabla c) + \nabla \cdot (\vec{v}\rho c) - s = 0$$

Since c is generally particles, energy etc, we can assume it is a scalar. Also if the density and diffusion coefficients are constant then the coefficients a_{ij} are:

$$a_{11} = a_{22} = a_{33} = -\mathcal{D}\rho$$
$$a_{44} = 0$$
$$\mathrm{Also\ all\ non-diagonal\ terms\ are\ zero}$$

The matrix created with a_{ij} is

$$M = \begin{bmatrix} -\mathcal{D}\rho & 0 & 0 & 0 \\ 0 & -\mathcal{D}\rho & 0 & 0 \\ 0 & 0 & -\mathcal{D}\rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues for this matrix are:

$$\lambda_1 = -\mathcal{D}\rho$$
$$\lambda_2 = -\mathcal{D}\rho$$
$$\lambda_2 = -\mathcal{D}\rho$$
$$\lambda_4 = 0$$

Hence the equation is parabolic

PROBLEM 3

Consider the system of uniaxial rods as shown below. We have three rods with Youngs' modulus and cross sectional areas E_e, A_e and lengths ℓ_e for $e = 1 \ldots 3$. Each element is labeled by $\boxed{e}$, $e = 1 \ldots 3$. Element $\boxed{2}$ and $\boxed{3}$ are connected via a rigid link at node $\textcircled{3}$, hence the j node of element $\boxed{2}$ and $\boxed{3}$ share the same global degree of freedom (see below).

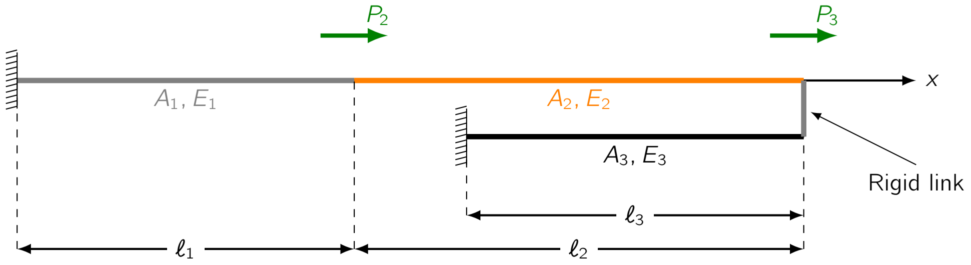

Figure 1: The system of uniaxial rods

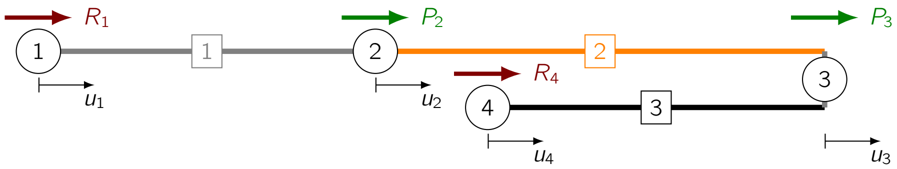

Figure 2: The system of uniaxial rods with nodes $\textcircled{n}$ and elements $\boxed{e}$ labeled.

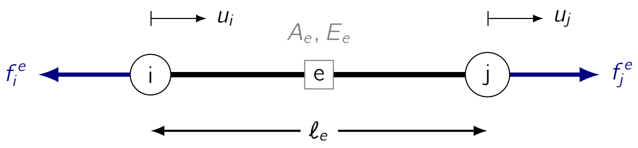

Figure 3: The local element view. Each element $\boxed{e}$ is composed of two nodes, node i and j. Their positions are denoted by x_i and x_j, $x_i < x_j$. The internal forces at node i and j for element e are denoted by $f_{i,j}^e$.

1. Write down the connectivity matrix. Namely, if we let each element be composed of two nodes (cf. Fig 3), node i and j with position x_i, x_j such that $x_i < x_j$, then identify the global node number corresponding to the each one of the local nodes. In short, fill in the table below.

Table 1: Connectivity array

element	i node	j node

Solution :

Table 2: Connectivity Array

element	i node	j node
1	1	2
2	2	3
3	4	3

2. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.

Solution :

$$\begin{bmatrix} -f_i^1 \\ f_j^1 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{\ell_1} & -\frac{A_1E_1}{\ell_1} \\ -\frac{A_1E_1}{\ell_1} & \frac{A_1E_1}{\ell_1} \end{bmatrix} \begin{bmatrix} u_i^1 \\ u_j^1 \end{bmatrix} \qquad \begin{bmatrix} -f_i^2 \\ f_j^2 \end{bmatrix} = \begin{bmatrix} \frac{A_2E_2}{\ell_2} & -\frac{A_2E_2}{\ell_2} \\ -\frac{A_2E_2}{\ell_2} & \frac{A_2E_2}{\ell_2} \end{bmatrix} \begin{bmatrix} u_i^2 \\ u_j^2 \end{bmatrix}$$

$$\begin{bmatrix} -f_i^3 \\ f_j^3 \end{bmatrix} = \begin{bmatrix} \frac{A_3E_3}{\ell_3} & -\frac{A_3E_3}{\ell_3} \\ -\frac{A_3E_3}{\ell_3} & \frac{A_3E_3}{\ell_3} \end{bmatrix} \begin{bmatrix} u_i^3 \\ u_j^3 \end{bmatrix}$$

3. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.

Solution :

$$\begin{bmatrix} -f_i^1 \\ f_j^1 \end{bmatrix} = \begin{bmatrix} \frac{A_1E_1}{\ell_1} & -\frac{A_1E_1}{\ell_1} \\ -\frac{A_1E_1}{\ell_1} & \frac{A_1E_1}{\ell_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{bmatrix} -f_i^2 \\ f_j^2 \end{bmatrix} = \begin{bmatrix} \frac{A_2E_2}{\ell_2} & -\frac{A_2E_2}{\ell_2} \\ -\frac{A_2E_2}{\ell_2} & \frac{A_2E_2}{\ell_2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} -f_i^3 \\ f_j^3 \end{bmatrix} = \begin{bmatrix} \frac{A_3E_3}{\ell_3} & -\frac{A_3E_3}{\ell_3} \\ -\frac{A_3E_3}{\ell_3} & \frac{A_3E_3}{\ell_3} \end{bmatrix} \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

4. For each node write the equilibrium equations in terms of the external forces P_k, $k = 2, 3$, the reactions R_k, $k = 1, 4$, and the internal forces $f_{i,j}^e$.

Solution :

$$\begin{aligned} R_1 &= -f_i^1 \\ P_2 &= f_j^1 - f_i^2 \\ P_3 &= f_j^2 + f_j^3 \\ R_4 &= -f_i^3 \end{aligned}$$

5. Let $k_i = A_iE_i/\ell_i$ for $i = 1 \dots 3$. Write down the equilibrium equations in matrix form. Namely, as we did in class, write the equilibrium equations with a load vector containing reactions and external forces,

denoted it by {P}, the stiffness matrix denoted by $[K]$, and the vector of displacements $\{U\}$ such that

$$[K]\{U\} = \{P\}.$$

Solution :

$$k_1 = \frac{A_1 E_1}{\ell_1} \qquad k_2 = \frac{A_2 E_2}{\ell_2} \qquad k_3 = \frac{A_3 E_3}{\ell_3}$$

$$\begin{bmatrix} R_1 \\ P_2 \\ P_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} -f_i^1 \\ f_j^1 - f_i^2 \\ f_j^2 + f_j^3 \\ -f_i^3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

6. Let $u_1 = a$, $u_4 = b$ with a, b being given quantities. Apply the boundary conditions to $[K], \{P\}$.

Solution :

$$\begin{bmatrix} a \\ P_2 \\ P_3 \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

PROBLEM 4

Consider the system of rods of Problem 3. Let $E_1 A_1 = 2$, $E_2 A_2 = 5$, $E_3 A_3 = 7$ and $\ell_1 = 5$, $\ell_2 = 10$, $\ell_3 = 7$ such that $x_1 = 0$, $x_2 = 5$, $x_3 = 15$, $x_4 = 8$.

1. Write in `python` a function `local_to_global_dof` that takes as arguments (1) the connectivity array, (2) the element number, and (3) the local degree of freedom (i or j) and returns the corresponding global degree of freedom. Namely:

```
local_to_global_dof( connectivity_array, element_number, local_dof )
```

```
Solution :
"""
Solves the python problem for the homework #2
"""
import numpy as np
import sympy as sp
import numpy.linalg as LA

# Define the material and geometrical properties
E = [2.0, 5.0, 7.0]
A = [1.0, 1.0, 1.0]
l = [5.0, 10.0, 7.0]
# Define the connectivity matrix
connectivity = np.array([[0,1],[1,2], [3,2]])
# Define the coordinates of the nodes
coordinates = [0.0,5.0,15.0,8.0 ]

# Function to return the global degree of freedom from the local degree of freedom
def local_to_global_dof(connectivity_array,element_number,local_dof):
        return connectivity_array[element_number,local_dof]
```

2. Write in `python` a function `element_stiffness` that takes as arguments (1) the element Young's modulus, (2) the element cross sectional area, (3) the x_i coordinate, (4) the x_j coordinate, and returns the element stiffness. Namely:

```
element_stiffness( youngs_modulus, area , x_i, x_j )
```

```
Solution :
# Function to return the element stiffness matrix
def element_stiffness(young_modulus, area, x_i, x_j):
        K_e = young_modulus*area/(x_j-x_i)*np.array([[1,-1],[-1,1]])
        return K_e
```

3. Write a loop that assembles the global element stiffness matrix. As aid you can (don't have to) follow the following pseudocode.

```
# Create arrays of material properties
youngs_moduli ← [ E_1, ...]

areas ← [A_1, ...]

# Create the connectivity array
connectivity_array ← [[global_i, global_j], ... ]

# Create the coordinates array
coordinates_array ← [x_1,x_2, ... ]

# Initialize the global stiffness matrix
K ← a zero matrix of size num_dof × num_dof

# Loop over all elements
for each element e

    x_{i,j} ← get the crds of the i and j nodes of element e

    youngs_e, area_e ← get properties of element e

    ke ← element_stiffness( youngs_modulus_e, area_e , x_i, x_j )

    for each local dof p

        global_p ← local_to_global_dof( connectivity_array, e, p )

        for each local dof q

            global_q ← local_to_global_dof( connectivity_array, e, q )

            K[ global_p, global_q ] += ke[p,q]
```

```
Solution :
# Function to return the global degree of freedom from the local degree of freedom
def local_to_global_dof(connectivity_array,element_number,local_dof):
        return connectivity[element_number,local_dof]
# Function to return the element stiffness matrix
def element_stiffness(young_modulus, area, x_i, x_j):
        K_e = young_modulus*area/(x_j-x_i)*np.array([[1,-1],[-1,1]])
        return K_e
# Initialize the global stiffness matrix
KG = np.zeros((num_dof,num_dof))
# Loop over all elements
for e in range(nel):
        x_i = coordinates[connectivity[e,0]] # The i coordinate of the element
        x_j = coordinates[connectivity[e,1]] # The j coordinate of the element
        E_e = E[e] # The young's modulus of the element
        A_e = A[e] # The area of the element
        K_e = element_stiffness(E_e,A_e,x_i,x_j) # Obtain the element stiffness matrix
        # Assemble the global stiffness matrix
        for p in range(ele_dof):
                global_p = local_to_global_dof(connectivity,e,p)
                for q in range(ele_dof):
                        global_q = local_to_global_dof(connectivity,e,q)
                        KG[global_p,global_q] += K_e[p,q]
```

4. As before assume u_1 and u_4 are given, namely $u_1 = 0, u_4 = 1$. Apply this boundary conditions to $[K]$ and $\{P\}$ where $P_2 = 10$, $P_3 = 2$.

```
Solution :
# Given load
P = np.zeros(num_dof)
P[1] = 10.0
P[2] = 2.0
# Nodes of known displacement
# Set one if known else 0
bc = [1,0,0,1]
# Dirichlet Boundary conditions
g = np.zeros(num_dof)
g[0] = 0.0
g[3] = 1.0
# Updated force matrix
F = np.zeros(num_dof)
# Initialize a new matrix with KG values
K = KG.copy()
# Updated Stiffness matrix
for b in range(len(bc)):
        for num in range(num_dof):
                if bc[b] == 1:
                        if b == num:
                                K[b,num] = 1.0
                        else:
                                K[b,num] = 0.0

# Updated Force matrix
for b in range(len(bc)):
        if bc[b] == 1:
                F[b] = g[b]
        else:
                F[b] = P[b]
```

5. What are the displacements of the nodes ?

```
Solution :
u = np.dot(LA.inv(K),F.T) # take the inverse and multiply
```

$$u_2 = 15$$
$$u_3 = 7$$

6. What are the reactions R_1, R_4 ?

```
Solution :
# Find reactions:
R1 = np.dot(KG[:,0],u)
R4 = np.dot(KG[:,3],u)
```

$$R_1 = -6$$
$$R_4 = -6$$