MID-TERM PRACTICE QUESTIONS

CEE 361-513: Introduction to Finite Element Methods

Thurday Oct. 19

This are <u>some</u> example questions to sharpen your skills for the mid-term. In addition you should review the homework, precepts, and lecture notes, as well as Chapter 1.1 - 1.9 and 1.12 - 1.14 of the Hughes book.

PROBLEM 1

1. Let $d = 2$. $u = x_1 x_2 + c$ be a scalar where c is any arbitrary constant. Find ∇u and $\nabla \cdot (\nabla u)$.

   Solution :

$$\nabla u = \frac{du}{dx_i}\mathbf{e}_i$$
$$= x_2\mathbf{e}_1 + x_1\mathbf{e}_2$$

$$\nabla \cdot (\nabla u) = \frac{d(\nabla u)}{dx_i} \cdot \mathbf{e}_i$$
$$= 0$$

2. Let $d = 3$. $\boldsymbol{u} = x_1 x_3 \mathbf{e}_1 + x_2 x_3 \mathbf{e}_2$. Find the gradient of $\boldsymbol{u}$.

   Solution :

$$\nabla \boldsymbol{u} = \frac{d\boldsymbol{u}}{dx_i} \otimes \mathbf{e}_i$$
$$= x_3\mathbf{e}_1 \otimes \mathbf{e}_1 + x_3\mathbf{e}_2 \otimes \mathbf{e}_2 + x_1\mathbf{e}_1 \otimes \mathbf{e}_3 + x_2\mathbf{e}_2 \otimes \mathbf{e}_3$$

3. Is $\boldsymbol{T}(\boldsymbol{u}) = \sin(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + \cos(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1$ a tensor?

   Solution :

$$\boldsymbol{T}(\alpha\boldsymbol{u}) = sin(\alpha\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + cos(\alpha\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1$$
$$\neq \alpha(sin(\boldsymbol{u} \cdot \mathbf{e}_1)\mathbf{e}_2 + cos(\boldsymbol{u} \cdot \mathbf{e}_2)\mathbf{e}_1)$$

   Hence not a tensor.

4. Let $\boldsymbol{x}_a = 2\mathbf{e}_1 + 5\mathbf{e}_2$ and $\boldsymbol{x}_b = 7\mathbf{e}_1 + 8\mathbf{e}_2$. Find the projection tensor that projects vectors along the direction $\boldsymbol{a} = \boldsymbol{x}_b - \boldsymbol{x}_a$.

Solution :

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{x}_b - \boldsymbol{x}_a \\ &= 5\mathbf{e}_1 + 3\mathbf{e}_2\end{aligned}$$

Unit vector along $\boldsymbol{a}$

$$\begin{aligned}\boldsymbol{n} &= \frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\ &= \frac{1}{\sqrt{34}}(5\mathbf{e}_1 + 3\mathbf{e}_2)\end{aligned}$$

Projection tensor:

$$\begin{aligned}\boldsymbol{T} &= \boldsymbol{n} \otimes \boldsymbol{n} \\ &= \frac{1}{34}\left(25\mathbf{e}_1 \otimes \mathbf{e}_1 + 15\mathbf{e}_1 \otimes \mathbf{e}_2 + 15\mathbf{e}_2 \otimes \mathbf{e}_1 + 9\mathbf{e}_2 \otimes \mathbf{e}_2\right)\end{aligned}$$

5. Let $\{\mathbf{e}_i\}_{i=1}^3$ be a set of orthonormal basis. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}$ be three vectors such that $\boldsymbol{b} = \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{a}\,\boldsymbol{a}/\|\boldsymbol{a}\|^2$. Show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent (i.e. $\alpha\boldsymbol{b} + \boldsymbol{a} = 0 \Rightarrow \alpha = 0$).

Solution :
Let $\alpha_1\boldsymbol{b}$ + $\alpha_2\boldsymbol{a}$ = 0 for two arbitrary values of α_1 and α_2. If they are linearly dependent.Taking the dot product with $\boldsymbol{a}$

$$\begin{aligned}&\boldsymbol{a} \cdot (\alpha_1\boldsymbol{b} + \alpha_2\boldsymbol{a}) = 0 \\ &\boldsymbol{a} \cdot \alpha_1\left(\boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{a}\ \frac{\boldsymbol{a}}{||\boldsymbol{a}||^2}\right) + \alpha_2\boldsymbol{a} \cdot \boldsymbol{a} = 0 \\ &\alpha_2||\boldsymbol{a}||^2 = 0\end{aligned}$$

Since this is true for any arbitrary α_2.

$$\alpha_2 = 0$$

Substituing in the original assumption leads to:

$$\alpha_1 = 0$$

Hence, $\boldsymbol{b}$ and $\boldsymbol{a}$ are linearly independent.

6. Let $\mathrm{tr}(\boldsymbol{A}) := \boldsymbol{A} : \mathbf{1}$ be the trace of a tensor $\boldsymbol{A}$. If $f = x_2x_3 + x_1x_3 + x_1x_2$, in which $\{\mathbf{e}_i\}_{i=1}^3$ is a set of orthonormal basis associated with the cartesian coordinates $\{x_i\}_{i=1}^3$. Show that $\nabla \cdot (\nabla f) = \mathrm{tr}(\nabla(\nabla(f)))$.

Solution :

$$\nabla \cdot (\nabla f) = \nabla \cdot ((x_3 + x_2)\mathbf{e}_1 + (x_3 + x_1)\mathbf{e}_2 + (x_2 + x_1)\mathbf{e}_3)$$
$$= 0$$

$$tr(\nabla(\nabla(f))) = tr(\nabla((x_3 + x_2)\mathbf{e}_1 + (x_3 + x_1)\mathbf{e}_2 + (x_2 + x_1)\mathbf{e}_3))$$
$$= tr(\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_1 \times \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_3)$$
$$= (\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_3) : \mathbf{1}$$
$$= 0$$

Hence they are equal.

PROBLEM 2

Consider the truss shown below. Foreach node we have associated coordinates $\boldsymbol{q}_z$ and associated global degrees of freedom $\boldsymbol{u}_z$, where both $\boldsymbol{q}$ and $\boldsymbol{u}$ are vectors. All elements have the same E, A.

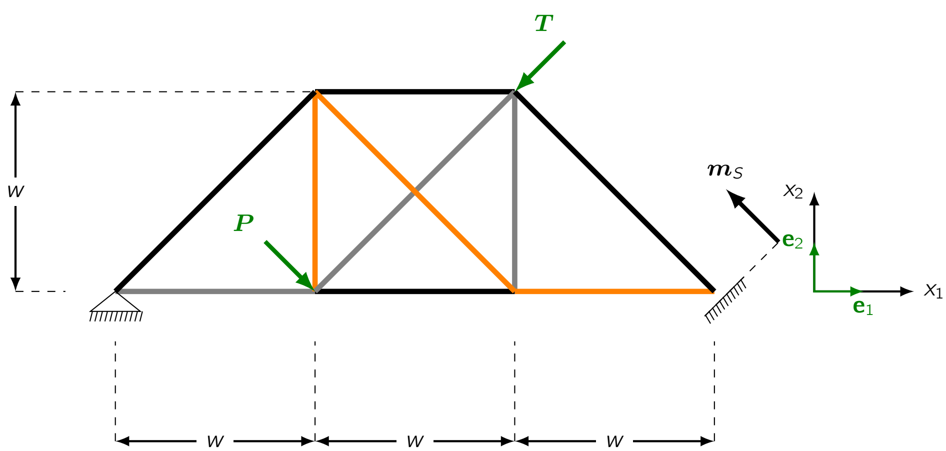


Figure 1: The system of uniaxial rods

1. Label each node and element and create a connectivity array.

Solution :

element	i node	j node
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	1
7	2	6
8	2	5
9	3	6
10	3	5

Table 1: Connectivity Array

2. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.

Solution :
For each element the element stiffness matrix could be found as:

$$\boldsymbol{K}_\mathrm{e} = \begin{bmatrix} \boldsymbol{k}_\mathrm{e} & \boldsymbol{k}_\mathrm{e} \\ \boldsymbol{k}_\mathrm{e} & \boldsymbol{k}_\mathrm{e} \end{bmatrix}$$

where $\boldsymbol{k}_\mathrm{e}$ is given as:

$$\boldsymbol{k}_\mathrm{e} = \frac{A_e E_e}{\ell_e} \boldsymbol{n} \otimes \boldsymbol{n}$$
$$\boldsymbol{n} = \frac{\boldsymbol{q}_j^\mathrm{e} - \boldsymbol{q}_i^\mathrm{e}}{|(\boldsymbol{q}_j^\mathrm{e} - \boldsymbol{q}_i^\mathrm{e})|}$$

The internal forces as a matrix vector operation could be written as:

$$\begin{bmatrix} -\boldsymbol{f}_i^e \\ \boldsymbol{f}_j^e \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}_e & -\boldsymbol{k}_e \\ -\boldsymbol{k}_e & \boldsymbol{k}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i^e \\ \boldsymbol{u}_j^e \end{bmatrix}$$

3. For each element write the internal forces as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.

Solution :
Internal forces in terms of global degree of freedoms for the first two elements:

$$\begin{bmatrix} -\boldsymbol{f}_i^1 \\ \boldsymbol{f}_j^1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}_1 & -\boldsymbol{k}_1 \\ -\boldsymbol{k}_1 & \boldsymbol{k}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} \qquad \begin{bmatrix} -\boldsymbol{f}_i^2 \\ \boldsymbol{f}_j^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}_2 & -\boldsymbol{k}_2 \\ -\boldsymbol{k}_2 & \boldsymbol{k}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{bmatrix}$$

Similarly others could be written using the connectivity array.

4. For each node write the equilibrium equations in terms of the external forces, the reactions, and the internal forces.

Solution :

$$
\begin{aligned}
\boldsymbol{R}_1 &= -\boldsymbol{f}_i^1 + \boldsymbol{f}_j^6 \\
\boldsymbol{P}_2 &= \boldsymbol{f}_j^1 - \boldsymbol{f}_i^2 - \boldsymbol{f}_i^7 - \boldsymbol{f}_i^8 \\
\boldsymbol{P}_3 &= \boldsymbol{f}_j^2 - \boldsymbol{f}_i^3 - \boldsymbol{f}_i^9 - \boldsymbol{f}_i^{10} \\
\boldsymbol{P}_4 &= \boldsymbol{f}_j^3 - \boldsymbol{f}_i^4 \\
\boldsymbol{P}_5 &= -\boldsymbol{f}_i^5 + \boldsymbol{f}_j^4 + \boldsymbol{f}_j^{10} + \boldsymbol{f}_j^8 \\
\boldsymbol{P}_6 &= \boldsymbol{f}_j^5 - \boldsymbol{f}_i^6 + \boldsymbol{f}_j^7 + \boldsymbol{f}_j^9
\end{aligned}
$$

5. Write down the equilibrium equations in matrix form. Namely, as we did in class, write the equilibrium equations with a load vector containing reactions and external forces, denoted it by {P}, the stiffness matrix denoted by $[K]$, and the vector of displacements $\{U\}$ such that

$$[K]\{U\} = \{P\}.$$

Solution :

$$
\begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \end{bmatrix}
=
\begin{bmatrix}
\boldsymbol{k}_1+\boldsymbol{k}_6 & -\boldsymbol{k}_1 & & & & -\boldsymbol{k}_6 \\
-\boldsymbol{k}_1 & \boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_7+\boldsymbol{k}_8 & -\boldsymbol{k}_2 & & -\boldsymbol{k}_8 & -\boldsymbol{k}_7 \\
 & -\boldsymbol{k}_2 & \boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_9+\boldsymbol{k}_{10} & -\boldsymbol{k}_3 & -\boldsymbol{k}_{10} & -\boldsymbol{k}_9 \\
 & & -\boldsymbol{k}_3 & \boldsymbol{k}_3+\boldsymbol{k}_4 & -\boldsymbol{k}_4 & \\
 & -\boldsymbol{k}_8 & -\boldsymbol{k}_{10} & -\boldsymbol{k}_4 & \boldsymbol{k}_4+\boldsymbol{k}_5+\boldsymbol{k}_8+\boldsymbol{k}_{10} & -\boldsymbol{k}_5 \\
-\boldsymbol{k}_6 & -\boldsymbol{k}_7 & -\boldsymbol{k}_9 & & -\boldsymbol{k}_5 & \boldsymbol{k}_5+\boldsymbol{k}_6+\boldsymbol{k}_7+\boldsymbol{k}_9
\end{bmatrix}
\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \end{bmatrix}
$$

6. At the leftmost node we prevent the truss from moving. At the rightmost node we allow the truss to move along a plane whose unit normal is $\boldsymbol{m}_2$. Apply the aforementioned conditions to $[K], \{P\}$.

Solution :

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{P}_2 \\ \boldsymbol{P}_3 \\ \boldsymbol{P}_4 \\ \boldsymbol{P}_5 \\ \boldsymbol{P}_6 \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \mathbf{0} \\ -\boldsymbol{k}_1 & \boldsymbol{k}_1+\boldsymbol{k}_2+ \\ \boldsymbol{k}_7+\boldsymbol{k}_8 & -\boldsymbol{k}_2 & & -\boldsymbol{k}_8 & -\boldsymbol{k}_7 & \mathbf{0} \\ & -\boldsymbol{k}_2 & \boldsymbol{k}_2+\boldsymbol{k}_3+ \\ \boldsymbol{k}_9+\boldsymbol{k}_{10} & -\boldsymbol{k}_3 & -\boldsymbol{k}_{10} & -\boldsymbol{k}_9 & \mathbf{0} \\ & & -\boldsymbol{k}_3 & \boldsymbol{k}_3+\boldsymbol{k}_4 & -\boldsymbol{k}_4 & & -\boldsymbol{m}_2 \\ & -\boldsymbol{k}_8 & -\boldsymbol{k}_{10} & -\boldsymbol{k}_4 & \boldsymbol{k}_4+\boldsymbol{k}_5+ \\ \boldsymbol{k}_8+\boldsymbol{k}_{10} & -\boldsymbol{k}_5 & \mathbf{0} \\ -\boldsymbol{k}_6 & -\boldsymbol{k}_7 & -\boldsymbol{k}_9 & & -\boldsymbol{k}_5 & \boldsymbol{k}_5+\boldsymbol{k}_6+ \\ \boldsymbol{k}_7+\boldsymbol{k}_9 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \boldsymbol{m}_2^T & \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \\ \lambda \end{bmatrix}$$

7. What is the reaction force at the leftmost node?

Solution :
All the dispalcements could be found after solving for the system. Once we have the dispalcements, we can find the reactions. Reaction force at the left-most node.

$$\boldsymbol{R}_1 = \begin{bmatrix} \boldsymbol{k}_1+\boldsymbol{k}_6 & -\boldsymbol{k}_1 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{k}_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \\ \lambda \end{bmatrix}$$

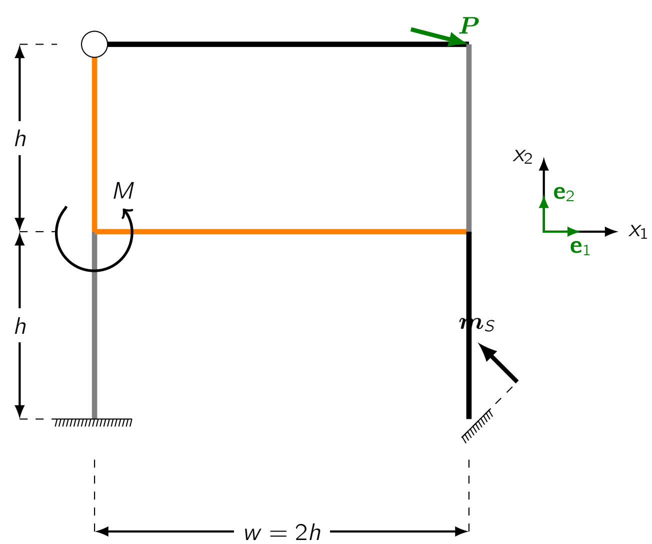
8. What is the reaction force at the rightmost nodes?

Solution :
All the dispalcements could be found after solving for the system. Once we have the dispalcements, we can find the reactions. Reaction force at the right-most node.

$$\boldsymbol{R}_4 = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{k}_3 & \boldsymbol{k}_3 + \boldsymbol{k}_4 & -\boldsymbol{k}_4 & \boldsymbol{O} & -\boldsymbol{m}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{u}_6 \\ \lambda \end{bmatrix}$$

PROBLEM 3

Consider the frame shown below. At the lower- and left-most node we constrain the frame from moving in all directions and we prevent it from rotating. At the upper- and left-most node we have a hinge (hence no moment can be transferred). At the lowest- and right-most support the frame is allowed to move along a plane define by the normal $\boldsymbol{m}_S$. All elements have the same E, I, A.



1. Label each element and node and write the connectivity array.

Solution :

element	i node	j node
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	2	5

Table 2: Connectivity Array

2. For each node write the equilibrium equations in terms of the external force $\boldsymbol{P}$ and moment M, and the internal forces $\boldsymbol{f}^e_{i,j}$ and moments $m^e_{i,j}$.

Solution :

$$
\begin{aligned}
\boldsymbol{V}_1 &= -\boldsymbol{f}_i^{n1} + \boldsymbol{f}_i^{s1}\\
M_1 &= -m_i^1\\
\boldsymbol{V}_2 &= \boldsymbol{f}_j^{n1} - \boldsymbol{f}_j^{s1} - \boldsymbol{f}_i^{n2} + \boldsymbol{f}_i^{s2} - \boldsymbol{f}_i^{n6} + \boldsymbol{f}_i^{s6}\\
M_2 &= -m_i^2 + m_j^1 - m_i^6\\
\boldsymbol{V}_3 &= \boldsymbol{f}_j^{n2} - \boldsymbol{f}_j^{s2} - \boldsymbol{f}_i^{n3} + \boldsymbol{f}_i^{s3}\\
M_3 &= m_j^2 - m_i^3\\
\boldsymbol{V}_4 &= \boldsymbol{f}_j^{n3} - \boldsymbol{f}_j^{s3} - \boldsymbol{f}_i^{n4} + \boldsymbol{f}_i^{s4}\\
M_4 &= m_j^3 - m_i^4\\
\boldsymbol{V}_5 &= \boldsymbol{f}_j^{n4} - \boldsymbol{f}_j^{s4} + \boldsymbol{f}_j^{n6} - \boldsymbol{f}_j^{s6} - \boldsymbol{f}_i^{n5} + \boldsymbol{f}_i^{s5}\\
M_5 &= -m_i^5 + m_j^4 + m_j^6\\
\boldsymbol{V}_6 &= \boldsymbol{f}_j^{n5} - \boldsymbol{f}_i^{s5}\\
M_1 &= m_j^5
\end{aligned}
$$

3. Write the general expression of internal forces (and moments) as the matrix vector operation of the *local element stiffness* and the *local degrees of freedom*.

Solution :
Internal forces as matrix vector operation of the local element stiffness and local degrees of freedom.

$$\begin{bmatrix} \boldsymbol{V}_i \\ M_i \\ \boldsymbol{V}_j \\ M_j \end{bmatrix} = \begin{bmatrix} [\boldsymbol{K_{fw}}] & [\boldsymbol{k_{f\theta}}] & [-\boldsymbol{K_{fw}}] & [\boldsymbol{k_{f\theta}}] \\ [\boldsymbol{k_{mw}}]^T & [k_{m\theta}] & [-\boldsymbol{k_{mw}}]^T & [\hat{k}_{m\theta}] \\ [-\boldsymbol{K_{fw}}] & [-\boldsymbol{k_{f\theta}}] & [\boldsymbol{K_{fw}}] & [-\boldsymbol{k_{f\theta}}] \\ [\boldsymbol{k_{mw}}]^T & [\hat{k}_{m\theta}] & [-\boldsymbol{k_{mw}}]^T & [k_{m\theta}] \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_i \\ \theta_i \\ \boldsymbol{w}_j \\ \theta_j \end{bmatrix}$$

where:

$$\boldsymbol{K_{fw}} = \frac{A_e E_e}{\ell_e} \boldsymbol{n}^e \otimes \boldsymbol{n}^e + \frac{12 E_e I_e}{\ell_e^3} \boldsymbol{s}^e \otimes \boldsymbol{s}^e$$
$$k_{m\theta} = \frac{4 E_e I_e}{\ell_e}$$
$$\hat{k}_{m\theta} = \frac{2 E_e I_e}{\ell_e}$$
$$\boldsymbol{k_{mw}} = \boldsymbol{k_{f\theta}} = \frac{6 E_e I_e}{\ell_e^2} \boldsymbol{s}^e$$

For each element e

4. For each element write the internal forces (and moments) as the matrix vector operation of the *local element stiffness* and the *GLOBAL degrees of freedom* using the connectivity array.

Solution :
For element 1

$$\begin{bmatrix} \boldsymbol{V}_1 \\ M_1 \\ \boldsymbol{V}_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{K}^1_{\boldsymbol{fw}}] & [\boldsymbol{k}^1_{\boldsymbol{f\theta}}] & [-\boldsymbol{K}^1_{\boldsymbol{fw}}] & [\boldsymbol{k}^1_{\boldsymbol{f\theta}}] \\ [\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & [k^1_{m\theta}] & [-\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & [\hat{k}^1_{m\theta}] \\ [-\boldsymbol{K}^1_{\boldsymbol{fw}}] & [-\boldsymbol{k}^1_{\boldsymbol{f\theta}}] & [\boldsymbol{K}^1_{\boldsymbol{fw}}] & [-\boldsymbol{k}^1_{\boldsymbol{f\theta}}] \\ [\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & [\hat{k}^1_{m\theta}] & [-\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & [k^1_{m\theta}] \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1 \\ \theta_1 \\ \boldsymbol{w}_2 \\ \theta_2 \end{bmatrix}$$

For element 2

$$\begin{bmatrix} \boldsymbol{V}_2 \\ M_2 \\ \boldsymbol{V}_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{K}^2_{\boldsymbol{fw}}] & [\boldsymbol{k}^2_{\boldsymbol{f\theta}}] & [-\boldsymbol{K}^2_{\boldsymbol{fw}}] & [\boldsymbol{k}^2_{\boldsymbol{f\theta}}] \\ [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & [k^2_{m\theta}] & [-\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & [\hat{k}^2_{m\theta}] \\ [-\boldsymbol{K}^2_{\boldsymbol{fw}}] & [-\boldsymbol{k}^2_{\boldsymbol{f\theta}}] & [\boldsymbol{K}^2_{\boldsymbol{fw}}] & [-\boldsymbol{k}^2_{\boldsymbol{f\theta}}] \\ [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & [\hat{k}^2_{m\theta}] & [-\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & [k^2_{m\theta}] \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_2 \\ \theta_2 \\ \boldsymbol{w}_3 \\ \theta_3 \end{bmatrix}$$

Ans similarly other elements could be written down.

5. Using $\boldsymbol{K}^e_{\boldsymbol{fw}}, \boldsymbol{k}^e_{\boldsymbol{f\theta}}, \ldots$, write down the equilibrium equations in matrix form.

Solution :

$$
\begin{bmatrix} \boldsymbol{V}_1 \\ M_1 \\ \boldsymbol{V}_2 \\ M_2 \\ \boldsymbol{V}_3 \\ M_3 \\ \boldsymbol{V}_4 \\ M_4 \\ \boldsymbol{V}_5 \\ M_5 \\ \boldsymbol{V}_6 \\ M_6 \end{bmatrix}
=
\begin{bmatrix}
\boldsymbol{K}^1_{\boldsymbol{fw}} & \boldsymbol{k}^1_{\boldsymbol{f\theta}} & -\boldsymbol{K}^1_{\boldsymbol{fw}} & \boldsymbol{k}^1_{\boldsymbol{f\theta}} & \boldsymbol{O} & \boldsymbol{0} & \dots & \dots \\
[\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & k^1_{m\theta} & -[\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & \hat{k}^1{}_{m\theta} & \boldsymbol{0}^T & 0 & \dots & \dots \\
-\boldsymbol{K}^1_{\boldsymbol{fw}} & -\boldsymbol{k}^1_{\boldsymbol{f\theta}} & \boldsymbol{K}^1_{\boldsymbol{fw}} + \boldsymbol{K}^2_{\boldsymbol{fw}} + \boldsymbol{K}^6_{\boldsymbol{fw}} & -\boldsymbol{k}^1_{\boldsymbol{f\theta}} + \boldsymbol{k}^2_{\boldsymbol{f\theta}} + \boldsymbol{k}^6_{\boldsymbol{f\theta}} & -\boldsymbol{K}^2_{\boldsymbol{fw}} & \boldsymbol{k}^2_{\boldsymbol{f\theta}} & \dots & \dots \\
[\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & \hat{k}^1{}_{m\theta} & [-\boldsymbol{k}^1_{\boldsymbol{mw}}]^T + [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T + [\boldsymbol{k}^6_{\boldsymbol{mw}}]^T & k^1_{m\theta} + k^2_{m\theta} + k^6_{m\theta} & -[\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & \hat{k}^2{}_{m\theta} & \dots & \dots \\
\boldsymbol{O} & \boldsymbol{0} & -\boldsymbol{K}^2_{\boldsymbol{fw}} & -\boldsymbol{k}^2_{\boldsymbol{f\theta}} & \boldsymbol{K}^2_{\boldsymbol{fw}} & -\boldsymbol{k}^2_{\boldsymbol{f\theta}} & \dots & \dots \\
\boldsymbol{0}^T & 0 & [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & \hat{k}^2{}_{m\theta} & [-\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & k^2_{m\theta} & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{bmatrix}
\begin{bmatrix} \boldsymbol{w}_1 \\ \theta_1 \\ \boldsymbol{w}_2 \\ \theta_2 \\ \boldsymbol{w}_3 \\ \theta_3 \\ \boldsymbol{w}_4 \\ \theta_4 \\ \boldsymbol{w}_5 \\ \theta_5 \\ \boldsymbol{w}_5 \\ \theta_5 \end{bmatrix}
$$

The matrix is incomplete. But this should get you started and you can fill the rest of the terms.

6. At the lower- and left-most node we constrain the frame from moving in all directions and we prevent it from rotating. At the upper- and left-most node we have a hinge (hence no moment can be transferred). At the lowest- and right-most support the frame is allowed to move along a plane define by the normal $\boldsymbol{m}_S$. Apply the aforementioned conditions to the matrix form of the previous step.

Solution :

We need to add an extra row and column to accomodate for the lagrange multiplier and update the boundary conditions.

$$
\begin{bmatrix} \boldsymbol{0} \\ 0 \\ \boldsymbol{V}_2 \\ M_2 \\ \boldsymbol{V}_3 \\ M_3 \\ \boldsymbol{V}_4 \\ M_4 \\ \boldsymbol{V}_5 \\ M_5 \\ \boldsymbol{V}_6 \\ M_6 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{0} & \dots & \dots \\
0^T & 1 & 0^T & 0 & \boldsymbol{0}^T & 0 & \dots & \dots \\
-\boldsymbol{K}^1_{\boldsymbol{fw}} & -\boldsymbol{k}^1_{\boldsymbol{f\theta}} & \boldsymbol{K}^1_{\boldsymbol{fw}} + \boldsymbol{K}^2_{\boldsymbol{fw}} + \boldsymbol{K}^6_{\boldsymbol{fw}} & -\boldsymbol{k}^1_{\boldsymbol{f\theta}} + \boldsymbol{k}^2_{\boldsymbol{f\theta}} + \boldsymbol{k}^6_{\boldsymbol{f\theta}} & -\boldsymbol{K}^2_{\boldsymbol{fw}} & \boldsymbol{k}^2_{\boldsymbol{f\theta}} & \dots & \dots \\
[\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & \hat{k}^1{}_{m\theta} & [-\boldsymbol{k}^1_{\boldsymbol{mw}}]^T + [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T + [\boldsymbol{k}^6_{\boldsymbol{mw}}]^T & k^1_{m\theta} + k^2_{m\theta} + k^6_{m\theta} & -[\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & \hat{k}^2{}_{m\theta} & \dots & \dots \\
\boldsymbol{O} & \boldsymbol{0} & -\boldsymbol{K}^2_{\boldsymbol{fw}} & -\boldsymbol{k}^2_{\boldsymbol{f\theta}} & \boldsymbol{K}^2_{\boldsymbol{fw}} & -\boldsymbol{k}^2_{\boldsymbol{f\theta}} & \dots & \dots \\
\boldsymbol{0}^T & 0 & [\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & \hat{k}^2{}_{m\theta} & [-\boldsymbol{k}^2_{\boldsymbol{mw}}]^T & k^2_{m\theta} & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & -\boldsymbol{m}_s \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \boldsymbol{m}^t_s & \dots
\end{bmatrix}
\begin{bmatrix} \boldsymbol{w}_1 \\ \theta_1 \\ \boldsymbol{w}_2 \\ \theta_2 \\ \boldsymbol{w}_3 \\ \theta_3 \\ \boldsymbol{w}_4 \\ \theta_4 \\ \boldsymbol{w}_5 \\ \theta_5 \\ \boldsymbol{w}_5 \\ \theta_5 \\ \lambda \end{bmatrix}
$$

Again the matrix is incomplete, but this should get you started.

7. How would you determine the reactions?

Solution :

Once we have found the dispalcement by solving the system updated with the boundary conditions, we can obtain the reactions by matrix-vector operation between the original global stiffness matrix and the now known degrees of freedom. So for example, if we are looking for reaction $\boldsymbol{R}_1$ and M_1, the matrix-vector operation would be:

$$\begin{bmatrix}\boldsymbol{R}_1\\ M_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{K}^1_{\boldsymbol{fw}} & \boldsymbol{k}^1_{\boldsymbol{f\theta}} & -\boldsymbol{K}^1_{\boldsymbol{fw}} & \boldsymbol{k}^1_{\boldsymbol{f\theta}} & \boldsymbol{O} & \boldsymbol{0} & \dots & \dots\\ [\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & k^1_{m\theta} & -[\boldsymbol{k}^1_{\boldsymbol{mw}}]^T & \hat{k}^1{}_{m\theta} & \boldsymbol{0}^T & 0 & \dots & \dots\end{bmatrix} \begin{bmatrix}\boldsymbol{w}_1\\ \theta_1\\ \boldsymbol{w}_2\\ \theta_2\\ \boldsymbol{w}_3\\ \theta_3\\ \boldsymbol{w}_4\\ \theta_4\\ \boldsymbol{w}_5\\ \theta_5\\ \boldsymbol{w}_5\\ \theta_5\\ \lambda\end{bmatrix}$$

PROBLEM 4

Consider the following strong form: find $u : (0,1) \to \mathbb{R}$ such that

$$-\frac{d^2u}{dx^2} + u + x^3 = 0, \quad \forall x \in (0,1)$$

For each of the following boundary conditions, state the set of trial and test functions and derive the weak form.

i. $u(0) = g_0$, $u(1) = g_1$

Solution :

The set of trial functions $\mathcal{S}$:

$$\mathcal{S} = \{u | u \in Smooth, u(0) = g_0, u(1) = g_1\}$$

The set of test functions $\mathcal{V}$:

$$\mathcal{V} = \{w | w \in Smooth, w(0) = 0, w(1) = 0\}$$

Multiplying both the sides by the weight w and integrating:

$$-\int_0^1 \frac{d^2u}{dx^2}\ w\ dx + \int_0^1 u\ w\ dx + \int_0^1 x^3 w\ dx = 0$$

$$-\frac{du}{dx}w\Big|_0^1 + \int_0^1 \frac{du}{dx}\frac{dw}{dx}\ dx + \int_0^1 u\ w\ dx + \int_0^1 x^3 w\ dx = 0$$

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}\ dx + \int_0^1 u\ w\ dx + \int_0^1 x^3 w\ dx = 0$$

ii. $\frac{du}{dx}(0) = h_0$, $u(1) = g_1$

Solution :

The set of trial functions $\mathcal{S}$:

$$\mathcal{S} = \{u | u \in Smooth, u(1) = g_1\}$$

The set of test functions $\mathcal{V}$:

$$\mathcal{V} = \{w | w \in Smooth, w(1) = 0\}$$

Multiplying both the sides by the weight w and integrating:

$$- \int_0^1 \frac{d^2u}{dx^2} \ w \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

$$- \frac{du}{dx} w \Big|_0^1 + \int_0^1 \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

$$h_0 w(0) + \int_0^1 \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

iii. $u(0) = g_0$, $\frac{du}{dx}(1) = h_1$

Solution :

The set of trial functions $\mathcal{S}$:

$$\mathcal{S} = \{u | u \in Smooth, u(0) = g_0\}$$

The set of test functions $\mathcal{V}$:

$$\mathcal{V} = \{w | w \in Smooth, w(0) = 0\}$$

Multiplying both the sides by the weight w and integrating:

$$- \int_0^1 \frac{d^2u}{dx^2} \ w \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

$$- \frac{du}{dx} w \Big|_0^1 + \int_0^1 \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

$$- h_1 w(1) + \int_0^1 \frac{du}{dx}\frac{dw}{dx} \ dx + \int_0^1 u \ w \ dx + \int_0^1 x^3 w \ dx = 0$$

PROBLEM 5

For the above BVP derive the matrix form and, assuming linear shape functions as shown in class,

i. Derive the *element* stiffness matrix

Solution :
For an element, the linear shape functions are given by:

$$\phi_1 = \frac{\xi_2 - \xi}{\xi_2 - \xi_1}$$
$$\phi_2 = \frac{\xi - \xi_1}{\xi_2 - \xi_1}$$

For us $\xi_1 = 0$ and $\xi_2 = 1$. The stiffness term for the above BVP is given by:

$$K_{ij} = \left(\int_0^1 \frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}\left(\frac{d\hat{x}}{d\xi}\right)^{-1} \; d\xi + \int_0^1 \phi_i \phi_j \frac{d\hat{x}}{d\xi} \; d\xi \right)$$

Substituting the value of ϕ_i and ϕ_j we obtain:

$$K^e = \begin{bmatrix} \frac{1}{h^e} + \frac{h^e}{3} & \frac{-1}{h^e} + \frac{h^e}{6} \\ \frac{-1}{h^e} + \frac{h^e}{6} & \frac{1}{h^e} + \frac{h^e}{3} \end{bmatrix}$$

where h^e is the length of the element.

ii. Assuming we discretize $(0, 1)$ into two elements, with the element stiffness matrix derived above, assemble the global stiffness matrix.

Solution :
Uisng the element stiffness matrix from above we have:

$$K^1 = \begin{bmatrix} \frac{13}{6} & \frac{-23}{12} \\ \frac{-23}{12} & \frac{13}{6} \end{bmatrix}$$
$$K^2 = \begin{bmatrix} \frac{13}{6} & \frac{-23}{12} \\ \frac{-23}{12} & \frac{13}{6} \end{bmatrix}$$

The Global stiffness matrix:

$$K = \begin{bmatrix} \frac{13}{6} & \frac{-23}{12} & 0 \\ \frac{-23}{12} & \frac{13}{3} & \frac{-23}{12} \\ 0 & \frac{-23}{12} & \frac{13}{6} \end{bmatrix}$$

PROBLEM 6

Consider the potential given by

$$\Pi[u] = \int_0^1 \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + \int_0^1 \frac{u^2}{2} dx + \int_0^1 x^3 u dx.$$

Find $\langle \delta\Pi, \delta u \rangle$.

Solution :

$$\langle \delta\Pi, \delta u \rangle = \left.\frac{d\Pi[u^*]}{d\alpha}\right|_{\alpha=0}$$

where $u^* = u + \alpha w$ where $w \in \mathcal{V}$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
\langle \delta\Pi, \delta u \rangle &= \left.\frac{d\Pi[u^*]}{d\alpha}\right|_{\alpha=0} \\
&= \frac{d}{d\alpha}\left.\left(\int_0^1 \frac{1}{2}\left(\frac{du^*}{dx}\right)^2 \; dx + \int_0^1 \frac{u^{*2}}{2} \; dx + \int_0^1 x^3 u^* \; dx\right)\right|_{\alpha=0} \\
&= \frac{d}{d\alpha}\left.\left(\int_0^1 \frac{1}{2}\left(\frac{d(u+\alpha w)}{dx}\right)^2 \; dx + \int_0^1 \frac{(u+\alpha w)^2}{2} \; dx + \int_0^1 x^3(u+\alpha w) \; dx\right)\right|_{\alpha=0} \\
&= \left.\left(\int_0^1 \left(\frac{d(u+\alpha w)}{dx}\right)\frac{dw}{dx} \; dx + \int_0^1 (u+\alpha w)w \; dx + \int_0^1 x^3 w \; dx\right)\right|_{\alpha=0} \\
&= \int_0^1 \frac{du}{dx}\frac{dw}{dx} \; dx + \int_0^1 uw \; dx + \int_0^1 x^3 w \; dx
\end{aligned}$$